INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 7163

Juin 2011

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7163--FR+ENG

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

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Asymptotic Expansion of Steady-State Potential in a High Contrast Medium with a Thin Resistive Layer

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Thème NUM — Systèmes numériques
Équipes-Projets MC2

Rapport de recherche n° 7163 — Juin 2011 — 26 pages

Abstract: We study the steady-state potential in a high contrast medium with a resistive thin layer. We provide asymptotic expansion of the potential at any order. Transmission conditions at any order and the corresponding variational formulation are given. We prove uniform estimates with respect to the thickness of the layer and with respect to its resistivity. The main insight consists in the uniform variational formulation whatever small the layer conductivity is. Numerical simulations illustrate the theoretical results.

Key-words: Asymptotic analysis, Finite Element Method, Laplace equations

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Conditions de transmission approchées pour des couches minces résistives.

Résumé :

Mots-clés : Analyse Asymptotique, Méthode des Eléments Finis, Equations de Laplace

Contents

1 Introduction

In the present paper the electro-quasistatic approximation of the Maxwell equations in a high contrast medium with an insulating thin layer is considered. We aim at providing asymptotic expansions at any order with respect to the membrane thickness, whatever small the conductivity of the layer is. The asymptotic for the thickness tending to zero are very different compared with the soft contrast medium [17, 18], since the effect of the thin layer appears at the zeroth order in the high contrast case.

1.1 Motivation

The distribution of the steady-state potential in a biological cell is important for bio-electromagnetic investigations. A sufficiently large amplitude of the transmembrane potential (TMP), which is the difference of the electric potentials between both sides of the cell membrane, leads to an increase of the membrane permeability [19, 23]. Molecules such as bleomycin can then diffuse across the plasma membrane. This phenomenon, called electropermeabilization, has been already used in oncology and holds promises in gene therapy [16, 22], justifying precise assessments of the TMP. Since the experimental measurements of the TMP on living cells are limited — mainly due to the membrane thinness, which is a few nanometers thick — a numerical approach is often chosen [19, 21]. However, these computations are confronted with the heterogeneous parameters of the biological cells. Therefore in this paper we derive a rigorous asymptotic analysis to tackle these numerical difficulties.

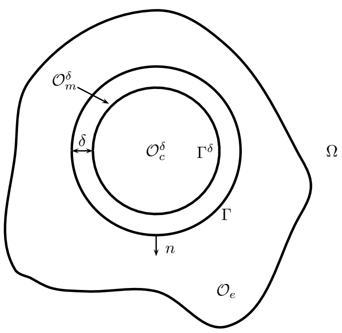
We consider the three-dimensional model of biological cell given by Schwan [13, 14] for different frequency ranges. This model considers the cell as a highly heterogeneous medium composed with a thin resistive membrane surrounding a conductive cytoplasm. Electro-quasistatic approximation[1] of the Maxwell equations in the time-harmonic regime is studied here. This approximation is usually considered to describe the behavior of a cell submitted to an electric field of frequency smaller than a few giga Hertz. Depending on the frequency, the modulus of the complex conductivity of the thin layer is either very small (for the frequency range under 10kHz), or of the same order (for the frequency range between 100kHz and 100MHz) compared with the membrane thickness δ, which is a small parameter.

At any order $k \in \mathbb{N}$ of accuracy δ^k, we aim at giving an asymptotic expansion of the potential, that is valid whatever the frequency is, and that avoids meshing the thin layer. More precisely, we provide uniform approximate transmission condition, that describes the effect of the layer without meshing it. This uniform estimate over the proposed range of frequency (up to 100MHz) enables us to apply safely the transmission conditions also for time-transient and non-linear problem, which are more relevant for modeling the electropermeabilization process [10].

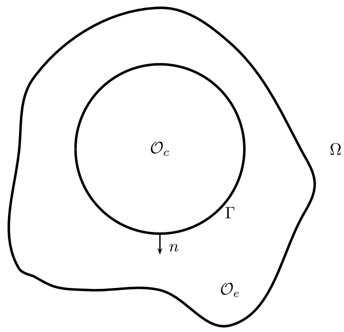
[1]The electro-quasistatic approximation consists in neglecting the curl part of the electric field, which therefore derives from a so-called electric potential. This amounts to considering the steady-state potential equations with complex coefficients.

1.2 The studied problem

The geometry of the problem is given in Fig. 1. We denote by Ω a bounded domain with smooth boundary $\partial\Omega$. Let $\mathcal{O}_c^\delta$ be a subdomain of Ω surrounded by a thin layer $\mathcal{O}_m^\delta$ with thickness δ. We assume that the domain $\mathcal{O}_m^\delta \cup \mathcal{O}_c^\delta$ is independent on δ and that its distance to $\partial\Omega$ is strictly positive. We denote by $\mathcal{O}_e = \Omega \setminus \overline{\mathcal{O}_m^\delta \cup \mathcal{O}_c^\delta}$. Observe that $\mathcal{O}_e$ is also δ-independent.



(a) The domain with a thin layer.



(b) The "background" domain.

Figure 1: Geometry of the problem.

Notation 1.1. *Present now the notations used throughout the paper.*

- *We generically denote by n the normal to a closed smooth surface of $\mathbb{R}^3$ outwardly directed from the domain enclosed by the surface (see Fig. 1) to the outer domain.*
- *Let $\mathcal{C}$ be a surface of $\mathbb{R}^3$, and let u be a function defined in a tubular neighborhood of $\mathcal{C}$. We define $u|_{\mathcal{C}^\pm}$ by*

$$\forall x \in \mathcal{C}, \quad u|_{\mathcal{C}^\pm}(x) = \lim_{t\to 0^+} u(x \pm tn(x)),$$

moreover if u is differentiable, we define $\partial_n u|_{\mathcal{C}^\pm}$ by

$$\forall x \in \mathcal{C}, \quad \partial_n u|_{\mathcal{C}^\pm}(x) = \lim_{t\to 0^+} \nabla u(x \pm tn(x)) \cdot n(x),$$

where $\cdot$ denotes the Euclidean scalar product of $\mathbb{R}^3$. In addition we define the jump $[u]_{\mathcal{C}}$ by

$$[u]_{\mathcal{C}} = u|_{\mathcal{C}^+} - u|_{\mathcal{C}^-}.$$

Let σ_c be the inner complex conductivity[2] of $\mathcal{O}_c^\delta$ and we denote similarly σ_e and σ_m the respective conductivities of $\mathcal{O}_e^\delta$ and $\mathcal{O}_m^\delta$. We suppose that both imaginary and real parts of σ_e, σ_c, and σ_m are positive. Let $\mathcal{O}_c$ be the smooth

[2]The complex conductivity σ of a given material is defined by

$$\sigma = \mathrm{i}\omega\varepsilon + s,$$

where s and ε denote respectively the (real) conductivity and permittivity of the material, and ω is the frequency of the time-harmonic field.

bounded domain defined by $\mathcal{O}_c = \Omega \setminus \overline{\mathcal{O}_e}$ and denote by Γ its boundary. We define the domain conductivity by

$$\sigma = \begin{cases} \sigma_c, \text{ in } \mathcal{O}_c^\delta, \\ \sigma_m, \text{ in } \mathcal{O}_m^\delta, \\ \sigma_e, \text{ in } \mathcal{O}_e, \end{cases} \quad \text{and } \tilde{\sigma} = \begin{cases} \sigma_c, \text{ in } \mathcal{O}_c, \\ \sigma_e, \text{ in } \mathcal{O}_e. \end{cases}$$

Throughout the paper the characteristic length of Ω as well as the characteristic conductivity of the domain are assumed equal to 1 so that we only deal with dimensionless quantities, however for the sake of simplicity we omit the term "dimensionless" that should be in front of each physical quantity.

The electro-quasistatic formulation is given by

$$\begin{cases} \nabla \cdot \left(\sigma \nabla u^\delta\right) = 0, \text{ in } \Omega \\ u^\delta|_{\partial\Omega} = g, \text{ on } \partial\Omega; \end{cases} \tag{1}$$

g is the electric potential imposed on the boundary of Ω. We suppose that g is as regular as we need. We aim at providing the rigorous asymptotic expansion of the potential u for δ tending to zero and $|\sigma_m| \leq \delta$.

Remark 1.2. *Multiply* (1) *by* $\overline{u^\delta}$ *and integrate by parts. There exists a constant* C *such that*

$$\|u^\delta\|_{H^1(\mathcal{O}_e)} \leq C|g|_{H^{1/2}(\partial\Omega)}, \quad \|\nabla u^\delta\|_{L^2(\mathcal{O}_c^\delta)} \leq C|g|_{H^{1/2}(\partial\Omega)},$$

and

$$\|\nabla u^\delta\|_{L^2(\mathcal{O}_m^\delta)} \leq \frac{C}{\sqrt{\sigma_m}}|g|_{H^{1/2}(\partial\Omega)}. \tag{2}$$

According to the last inequality, the more resistive the thin layer is, the worse the estimate of the electric field ∇u^δ *in the membrane is since* σ_m *is small for a resistive medium. Estimate* (2) *makes us think that "the electric field is trapped into the membrane". Therefore, unlike the case of a soft contrast medium, the resistive thin layer has an influence on the zeroth approximation of the potential.*

1.3 State of the art and plan of the paper

The main insight of the paper is to consider that the thin layer is highly insulating in the sense that

$$\sigma_m = \delta \xi S_m, \tag{3}$$

where $|S_m| = 1$ and $0 < |\xi| \leq 1$. Depending on the frequency of the studied phenomenon, the parameter $|\xi|$, which is a non-dimension frequency, can be of the order 1, δ, or even smaller. Observe that S_m can be seen as a dimensionless surface conductivity. For each frequency $\xi = \delta^q$, with $q \in \mathbb{N}$, it is possible to provide a precise asymptotic expansion by solving specific partial differential equations. However, in this paper we aim at giving a rigorous uniform variational formulation, that allows to approach the steady-state potential accurately and for all the different cases in a same manner.

Several papers in the mathematical research area are devoted to domains with thin layer. In [9] (see also the numerous works of Ammari *et al.* [4, 8, 3, 5, 6, 7]), Capdeboscq and Vogelius provide a general representation of the conductivity in domains with soft contrast small inclusions. They prove that the potential can be approached by the "background" potential —the potential in the domain without inclusion— with the help of the so-called polarization tensor. Their result proves that for a soft contrast small inclusion the influence of the inhomogeneity is the same as a dipole characterized by the polarization tensor (see Theorem 1. page 161 of [9]). More recently, Vogelius and Nguyen [24] show that this general representation formula holds for high contrast small diameter inclusion. They also mention that the representation formula does not hold in the case of high contrast thin inclusion. The aim of the present paper is to provide a precise description of the potential for such thin high resistive domains.

The method used to derive the effective transmission conditions has been extensively described previously in the case of soft contrast media. We refer to [12] for the original paper and to [18, 17] for a more general description. It consists in a suitable change of variables in the thin layer in order to make the small parameter appear in the equations. Recently Schmidt and Tordeux provided the asymptotic expansion of the solution to a bidimensional eddy-current like problem for thin conductive layers [20]. Our problem is different since the high contrast appears in the transmission conditions satisfied by the potential (that is not the case of Schmidt and Tordeux since the high contrast holds on the zeroth order operator of their second order partial differential equation). Moreover we consider the three-dimensional resistive case without any assumption on the magnitude of the layer conductivity. More precisely, we only suppose that $|\sigma_m|$ is smaller than the layer thickness δ whereas Schmidt and Tordeux choose their conductivity proportional to $1/\delta$. In this sense, our result is more general for the resistive case. Another difference lies in the fact that the solution to our limit problem does not belong to the Sobolev space H^1 but it is only piecewise H^1 (denoted by PH^1), and therefore an appropriate analysis have to be performed. Despite these differences we let the reader observe that in our result as in the result of Schmidt and Tordeux, the layer influence appears at the zeroth order term, which highlights the main insight of high contrast thin domains.

The outline of the paper is the following. In Section 2, we perform the change of variables, that will be useful to derive the asymptotic expansion. We then perform the formal identification of the coefficients in Section 3, and we provide preliminary estimates in Section 4. Section 5 is devoted to the proof of our main result, and in the last section, numerical simulations with the finite element method illustrate the theoretical results. We end by concluding remarks that present two applications of the results provided in this paper: the dynamic electric potential in biological cells, and the magnetostatic field in carbon nanotubes trapped in the wall of foam bubbles.

In the following subsections we present the model obtained by electrical engineers by physical considerations, and then we present our main result, that justify and extend the electrical engineering results.

1.4 Resistive thin layer in electrical engineering

Several papers in the electrical engineering research area have studied the electric potential in domains with thin resistive layers. We refer the reader to the survey of Bossavit [2] —and references therein— for the modelling principle. More recently, Pucihar *et al.* have proposed in [19] an electrical model of biological cells where the electric potential u^{approx} satisfies

$$\begin{cases} \Delta u^{\mathrm{approx}} = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^{\mathrm{approx}}|_{\partial\Omega} = g, \\ S_m \left[u^{\mathrm{approx}} \right]_\Gamma - \sigma_e \partial_n u^{\mathrm{approx}}|_{\Gamma^+} = 0, \\ \left[\tilde{\sigma} \partial_n u^{\mathrm{approx}} \right]_\Gamma = 0, \end{cases} \tag{4}$$

where S_m is the surface membrane conductivity with $\xi = 1$ in (3). Observe that in this model the membrane influence appears in the transmission condition. We aim at justifying the model of Pucihar *et al.* and at providing more accurate approximations of u^δ by performing a rigorous asymptotic expansion of the potential at any order, that are valid from the resistive case to the very resistive case.

An interesting result of the paper consists in the variational formulation of problem (4). More precisely, in their paper Pucihar *et al.* propose to solve problem (4) using an iterative method, that consists in making explicit the Neumann trace of the potential to compute the jump of the potential at the following iteration. For such a method several computations are needed before reaching an accurate value of the potential. This can be time– and resource– consuming, especially when studying a large frequency range, or when dealing with non-linear problems (in the electroporation modelling, the membrane conductivity depends on the TMP [1]). To avoid this drawback we propose a variational formulation that allows to solve the problem in only one computation. In addition, this formulation is useful in the justification of the asymptotics.

1.4.a The functional space $PH^1(\Omega)$

Denote by $PH^1(\Omega)$ the space of functions in Ω, which have the H^1–regularity in $\mathcal{O}_e$ and in $\mathcal{O}_c$ (but not in Ω) and let $PH^1_0(\Omega)$ be the subspace of functions belonging to $PH^1(\Omega)$ that vanish on $\partial\Omega$:

$$PH^1(\Omega) = \left\{ v : \quad v|_{\mathcal{O}_e} \in H^1(\mathcal{O}_e), \quad v|_{\mathcal{O}_c} \in H^1(\mathcal{O}_c) \right\}, \tag{5}$$

$$PH^1_0(\Omega) = \left\{ v : \quad v \in PH^1(\Omega), \quad v|_{\partial\Omega} = 0 \right\}. \tag{6}$$

For any function $g \in H^{1/2}(\partial\Omega)$ we define the space $g + PH^1_0(\Omega)$ by

$$g + PH^1_0(\Omega) = \left\{ v \in PH^1(\Omega) : v|_{\partial\Omega} = g \right\}. \tag{7}$$

1.4.b Variational formulation of problem (4)

The variational formulation of problem (4) satisfied by u^{approx} writes

$$\left\{\begin{array}{l}\text{Find } u^{\text{approx}} \in g + PH_0^1(\Omega) \text{ such that for all } \phi \in PH_0^1(\Omega)\\ \displaystyle\int_\Omega \tilde{\sigma}\nabla u^{\text{approx}} \cdot \nabla\phi \mathrm{d}x + \xi \int_\Gamma S_m [u^{\text{approx}}]_\Gamma [\phi]_\Gamma \mathrm{d}s = 0.\end{array}\right.$$

1.5 Main result

Before stating our main result we present the change of variables used in order to derive the asymptotic expansion.

1.5.a Geometry

We suppose that the inner boundary Γ of the domain $\mathcal{O}_e$ is a smooth compact 2D–manifold of $\mathbb{R}^3$ without boundary. Let $\mathbf{x_T} = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathbf{x_T})\}$. Define the map Φ by

$$\forall(\mathbf{x_T}, x_3) \in \Gamma \times \mathbb{R}, \quad \Phi(\mathbf{x_T}, x_3) = \psi(\mathbf{x_T}) + x_3 n(\mathbf{x_T}),$$

where n is the normal vector of Γ defined in Notation 1.1. The thin layer $\mathcal{O}_m^\delta$ is then parameterized by

$$\mathcal{O}_m^\delta = \{\Phi(\mathbf{x_T}, x_3), \quad (\mathbf{x_T}, x_3) \in \Gamma \times (-\delta, 0)\}.$$

The Euclidean metric in $(\mathbf{x_T}, x_3)$ is given by the 3×3–matrix $(g_{ij})_{i,j=1,2,3}$ where $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$:

$$\forall\alpha \in \{1,2\}, \quad g_{33} = 1,\ g_{\alpha 3} = g_{3\alpha} = 0, \tag{8a}$$

$$\forall(\alpha,\beta) \in \{1,2\}^2, \quad g_{\alpha\beta}(\mathbf{x_T}, x_3) = g^0_{\alpha\beta}(\mathbf{x_T}) + 2x_3 b_{\alpha\beta}(\mathbf{x_T}) + x_3^2 c_{\alpha\beta}(\mathbf{x_T}), \tag{8b}$$

where

$$g^0_{\alpha\beta} = \langle \partial_\alpha\psi, \partial_\beta\psi \rangle, \quad b_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta \psi \rangle, \quad c_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta n \rangle. \tag{8c}$$

We denote by (g^{ij}) the inverse matrix of (g_{ij}), and by g the determinant of (g_{ij}). For all $l \in \mathbb{N}$ define

$$\left\{\begin{array}{l} a^l_{ij} = \partial_3^l \left(\dfrac{\partial_i \left(\sqrt{g} g^{ij}\right)}{\sqrt{g}} \right)\Bigg|_{x_3=0}, \quad \text{for } (i,j) \in \{1,2,3\}^2, \\ A^l_{\alpha\beta} = \partial_3^l \left(g^{\alpha\beta}\right)\big|_{x_3=0}, \quad \text{for } (\alpha,\beta) \in \{1,2\}^2, \end{array}\right. \tag{9}$$

and let $\mathcal{S}^l_\Gamma$ be the differential operator on Γ of order 2 defined by

$$\mathcal{S}^l_\Gamma = \sum_{\alpha,\beta=1,2} a^l_{\alpha\beta}\partial_\beta + A^l_{\alpha\beta}\partial_\alpha\partial_\beta, \tag{10}$$

For all $q \geq 0$ the differential operators R^q_N and R^q_D are defined by the following recurrence formulae:

$$R^0_N = 0,\ R^0_D = \mathrm{Id}, \quad R^1_N = \mathrm{Id},\ R^1_D = 0, \tag{11a}$$

and for all $p \geq 0$

$$R_N^{p+2} = -\sum_{l=0}^{p} C_p^l \left(a_{33}^l R_N^{p+1-l} + R_N^{p-l} \mathcal{S}_\Gamma^l \right), \tag{11b}$$

$$R_D^{p+2} = -\sum_{l=0}^{p} C_p^l \left(a_{33}^l R_D^{p+1-l} + R_D^{p-l} \mathcal{S}_\Gamma^l \right), \tag{11c}$$

where $C_p^l = p!/(l!(p-l)!)$.

1.5.b Asymptotic at any order

The main result of the present paper holds in the following Theorem, that provides asymptotic expansion of the steady-state potential at any order.

Theorem 1.3 (Main result)**.** *Let g belong to $H^{5/2+n}(\partial\Omega)$, for $n \in \mathbb{N}$. The integer n denotes the order of the approximation.*

Let the four sequences $(g_D^p)_{0\leq p\leq n}$, $(g_N^p)_{0\leq p\leq n}$, $(u^p)_{-2\leq p\leq n}$ and $(u_m^p)_{-2\leq p\leq n}$ be defined by induction :

$$u^{-2} = u^{-1} = 0, \textit{ in } \Omega, \quad u_m^{-2} = u_m^{-1} = 0, \textit{ in } (-1,0)\times\mathbb{T}, \quad g_D^0 = g_N^0 = 0, \textit{ on } \Gamma. \tag{12a}$$

For $p \geq 0$ the function u^p is defined in Ω by

$$\Delta u^p = 0, \quad \textit{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^p|_{\partial\Omega} = \delta_0^p g, \textit{ where } \delta_0^p \textit{ is the classical Kronecker symbol,} \tag{12b}$$

$$\xi S_m \left[u^p\right]_\Gamma - \sigma_e \partial_n u^p|_{\Gamma^+} = g_D^p \tag{12c}$$

$$\left[\tilde{\sigma}\partial_n u^p\right]_\Gamma = g_N^p, \tag{12d}$$

the profile function u_m^p is defined in the cylinder $\Gamma \times (-1,0)$ by

$$u_m^p = u^p|_{\Gamma^+} + \eta \frac{\sigma_e}{\xi S_m} \partial_n u^p|_{\Gamma^+} + \int_\eta^0 (\eta - s) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{s^l}{l!} \left(\frac{s}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}s, \tag{12e}$$

and for $p \geq 1$ the function g_D^p and g_N^p are defined on Γ by

$$g_D^p = \xi S_m \sum_{l=1}^{p} \frac{1}{l!} \left(R_N^l(\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l(u^{p-l}|_{\Gamma^-}) \right)$$
$$+ (-1)^p \xi S_m \int_{-1}^{0} (1+\eta) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}\eta. \tag{13a}$$

$$g_N^p = \sigma_c \sum_{l=1}^{p} \frac{1}{l!} \Big(R_N^{l+1}(\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1}(u^{p-l}|_{\Gamma^-}) \Big)$$
$$- (-1)^p \xi S_m \int_{-1}^{0} \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}\eta. \tag{13b}$$

The above functions are uniquely determined and satisfy the following regularity for $0 \leq p \leq n$

$$g_D^p \in H^{5/2+n-p}(\Gamma), \quad g_N^p \in H^{3/2+n-p}(\Gamma), \tag{14}$$
$$u^p|_{\mathcal{O}_e} \in PH^{3+n-p}(\mathcal{O}_e), \quad u^p|_{\mathcal{O}_c} \in PH^{4+n-p}(\mathcal{O}_e), \tag{15}$$
$$u_m^p \in \mathscr{C}^\infty \Big((-1,0); H^{5/2+n-p}(\Gamma) \Big). \tag{16}$$

Denote by

$$u_{app}^n = \sum_{p=0}^{n} (-\delta)^p u^p,$$

then there exists a constant C_n independent on δ and ξ such that

$$\|u - u_{app}^n\|_{H^1(\mathcal{O}_e)} \leq C_n \xi \delta^{n+1} |g|_{H^{5/2+n}(\partial\Omega)}, \tag{17}$$

and for all ω compactly embedded in $\mathcal{O}_c$

$$\|u - u_{app}^n\|_{H^1(\omega)} \leq C_n \delta^{n+1} |g|_{H^{5/2+n}(\partial\Omega)}, \tag{18}$$
$$\|\nabla(u - u_{app}^n)\|_{L^2(\omega)} \leq C_n \xi \delta^{n+1} |g|_{H^{5/2+n}(\partial\Omega)}. \tag{19}$$

Observe that in the elementary problem at the order p, the source terms g_D^p and g_N^p depend on the solutions u^q and u_m^q of the elementary problems at the order q for $q = 0, \cdots, p-1$.

For $n = 0$, the approximate potential is the same as the potential of Pucihar *et al.*, therefore our result is a rigorous generalization of the electrical engineering models of resistive thin layers. We provide now the two first order terms of the expansion.

1.5.c Approximate transmission conditions at the zeroth and at the first orders

To obtain the two first order terms, we first observe that according to definition (10), $\mathcal{S}_\Gamma^0$ is the Laplace-Beltrami operator on Γ, therefore we denote it by

$\mathcal{S}_\Gamma^0 = \Delta_\Gamma$. Moreover the mean curvature $\mathscr{H}$ of the surface Γ equals

$$\mathscr{H} = \frac{a_{33}^0}{2}.$$

Hence, applying recurrence formula (12), we get the two first orders of the asymptotics.

The order 0

$$\begin{cases} \Delta u^0 = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \\ \left[\widetilde{\sigma}\partial_n u^0\right]_\Gamma = 0, \quad \xi S_m \left[u^0\right]_\Gamma - \sigma_e \partial_n u^0|_{\Gamma^+} = 0. \\ u^0|_{\partial\Omega} = g. \end{cases} \tag{20}$$

In the rescaled membrane $(-1, 0) \times \Gamma$

$$u_m^0 = \eta[u^0]_\Gamma + u^0|_{\Gamma^+}.$$

The order 1

$$\begin{cases} \Delta u^1 = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \\ \left[\widetilde{\sigma}\partial_n u^1\right]_\Gamma = -\sigma_c \Delta_\Gamma u^0|_{\Gamma^-}, \\ \xi S_m \left[u^1\right]_\Gamma - \sigma_e \partial_n u^1|_{\Gamma^+} = \xi S_m \left(1 - \mathscr{H}\dfrac{\sigma_c}{\xi S_m}\right) \partial_n u^0|_{\Gamma^-}. \\ u^1|_{\partial\Omega} = 0. \end{cases} \tag{21}$$

In the rescaled membrane $(-1, 0) \times \Gamma$

$$u_1^m = u^1|_{\Gamma^+} + \eta \frac{\sigma_e}{\xi S_m} \partial_n u^1|_{\Gamma^+} - \eta^2 \mathscr{H}[u^0]_\Gamma$$

Observe that at the effect of the geometry appears at the first order term, unlike the case of a soft contrast medium with a thin layer where it appears at the second order term [17, 18]. If we were considering an infinite cylinder, and still denoting by Γ the curve of the cross section, a_{33}^0 would be the curvature of Γ.

We recall that the zeroth order approximation is u^0 and the first order approximation is $u^0 - \delta u^1$. In Section 6 we illustrate numerically the theoretical results by studying the rate of convergence of the errors $u - u^0$ and $u - (u^0 - \delta u^1)$ in H^1–norm in the outer medium $\mathcal{O}_e$. We prove now the asymptotic expansion.

2 Preliminary results

2.1 Laplace operator for functions

The Laplace-Beltrami operator acting on functions for the metric $(g_{ij})_{i,j=1,2,3}$ is given by[3] [11]

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1,2,3} \partial_i \left(\sqrt{g} g^{ij} \partial_j\right). \tag{22}$$

[3] ∂_i denotes the partial derivative with respect to the variable x_i, for $i = 1, 2, 3$.

According to (8b) and using the smoothness and the compacity of Γ, the functions g_{ij} are analytic on the cylinder $\Gamma \times [-\delta, 0]$, for δ small enough. Hence the functions a_{ij}^l and $A_{\alpha\beta}^l$ defined by (9) are smooth functions defined on Γ. We infer the expansion of the Laplacian in local coordinates:

$$\Delta_g = \partial_3^2 + \sum_{l \geq 0} \frac{x_3^l}{l!} \left(a_{33}^l \partial_3 + \mathcal{S}_\Gamma^l \right), \qquad \forall (\mathbf{x_T}, x_3) \in \Gamma \times (-\delta, 0), \tag{23}$$

where the differential operators $\mathcal{S}_\Gamma^l$ are given by (10). Performing the change of variable $\eta = x_3/\delta$ we derive

$$\Delta_g = \frac{1}{\delta^2} \partial_\eta^2 + \frac{1}{\delta} a_{33}^0 \partial_\eta + \sum_{l \geq 0} \delta^l \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta + \mathcal{S}_\Gamma^l \right), \qquad \forall (\mathbf{x_T}, \eta) \in \Gamma \times (-1, 0). \tag{24}$$

The following proposition will be useful in order to derive the asymptotics.

Proposition 2.1. *Let u be a smooth harmonic function in $\mathcal{O}_c$. We still denote by u the function $u \circ \Phi$ in the tubular neighborhood of Γ. For all $p \in \mathbb{N}$ there exists two surfacic operators on Γ denoted by R_N^p and R_D^p respectively of order $p-1$ and p, and defined by* (11) *such that*

$$\partial_3^p u|_{\Gamma^-} = R_N^p(\partial_n u|_{\Gamma^-}) + R_D^p(u|_{\Gamma^-}). \tag{25}$$

Proof. The proof is an application of the Leibniz rule for the differentiation applied to (23). □

The above proposition leads to the following corollary by applying the Taylor formula with integral remainder term in the variable η.

Corollary 2.2. *Let u be a smooth harmonic function in $\mathcal{O}_c$, and denote by $\Gamma_{-\delta}$ the boundary of $\mathcal{O}_c^\delta$. There exists a sequence of operators $(T^p)_{p \geq 0}$ of order p such that for all $p \in \mathbb{N}$*

$$u|_{\Gamma_{-\delta}^-} = \sum_{k=0}^{p} \frac{(-\delta)^k}{k!} \left(R_N^k(\partial_n u|_{\Gamma^-}) + R_D^k(u|_{\Gamma^-}) \right) + \frac{(-\delta)^{p+1}}{(p+1)!} T^{p+1}(u). \tag{26}$$

$$\partial_n u|_{\Gamma_{-\delta}^-} = \sum_{k=0}^{p} \frac{(-\delta)^k}{k!} \left(R_N^{k+1}(\partial_n u|_{\Gamma^-}) + R_D^{k+1}(u|_{\Gamma^-}) \right) + \frac{(-\delta)^{p+1}}{(p+1)!} T^{p+2}(u), \tag{27}$$

where there exists a constant C such that for all $p \geq 0$

$$|T^{p+i}(u)|_{H^{1/2-i}(\Gamma)} \leq C \|u\|_{H^{p+2}(\mathcal{O}_c)}, \quad i = 0, 1.$$

We are now ready to derive formally the asymptotic expansion.

3 Formal asymptotics

Define the so-called profile function u_m in $\Gamma \times (-1, 0)$ by

$$\forall (\mathbf{x_T}, \eta) \in \Gamma \times (-1, 0), \quad u_m(\mathbf{x_T}, \eta) = u \circ \Phi(\mathbf{x_T}, \delta\eta).$$

Since $\sigma_m/\delta = \xi S_m$, the transmission conditions inherent to (1) write now

$$\sigma_c \partial_n u|_{\Gamma^-_{-\delta}} \circ \Phi(\bullet, -\delta) = \xi S_m \partial_\eta u_m|_{\eta=-1}, \quad u|_{\Gamma^-_{-\delta}} \circ \Phi(\bullet, -\delta) = u_m|_{\eta=-1}, \tag{28}$$

$$\sigma_e \partial_n u|_{\Gamma^+} \circ \psi = \xi S_m \partial_\eta u_m|_{\eta=0}, \quad u|_{\Gamma^+} \circ \psi = u_m|_{\eta=0}. \tag{29}$$

To obtain our transmission conditions, we set

$$\begin{cases} u = u^0 - \delta u^1 + \delta^2 u^2 - \cdots, \text{ in } \mathcal{O}_c^\delta \cup \mathcal{O}_e, \\ u_m = u_m^0 + \delta u_m^1 + \delta^2 u_m^2 + \cdots, \text{ in } \Gamma \times (-1, 0), \end{cases} \tag{30}$$

and we perform the identification of the terms with the same power in δ. Observe that the formal series for u is in $(-\delta)^k$ whereas the formal series of the profile terms involves δ^k. We choose this convention for the sake of simplicity since the δ^k appears naturally in the equality (24), while $(-\delta)^k$ appears in (26)–(27).

We emphasize that the coefficients u^j will be defined in the whole domain Ω even if $\sum_{j=0}^p (-\delta)^j u^j$ approximates u only in $\mathcal{O}_e \cup \mathcal{O}_c^\delta$. Observe that equations (28) are written on $\Gamma_{-\delta}$, while we want to write transmission conditions on Γ. Insert the ansatz (30) into equalities (26)–(27) of Corollary 2.2 to derive

$$u|_{\Gamma^-_{-\delta}} = \sum_{p\geq 0} (-\delta)^p \left(u^p|_{\Gamma^-} + \sum_{l=1}^p \frac{1}{l!} \Big(R_N^l (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l (u^{p-l}|_{\Gamma^-}) \Big) \right), \tag{31}$$

$$\partial_n u|_{\Gamma^-_{-\delta}} = \sum_{p\geq 0} (-\delta)^p \left(\partial_n u^p|_{\Gamma^-} + \sum_{l=1}^p \frac{1}{l!} \Big(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} (u^{p-l}|_{\Gamma^-}) \Big) \right). \tag{32}$$

3.1 Recurrence formulae

Let us now identify the terms with the same power in δ. We necessarily have

$$\Delta u^p = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^0|_{\partial\Omega} = \delta_0^p g.$$

Remind that δ_0^p is the Kronecker symbol equal to 1 if $p = 0$ and 0 otherwise. Using (24), write now the equations satisfied by u_m^p in $\Gamma \times (-1, 0)$:

$$\partial_\eta^2 u_m^p = -a_{33}^0 \partial_\eta u_m^{p-1} - \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right). \tag{33}$$

According to transmission conditions (28)–(29), by integrating relations (33) and (32), we infer

$$\xi S_m \partial_\eta u_m^p|_{\eta=-1} = (-1)^p \sigma_c \left(\partial_n u_{\Gamma^-}^p + \sum_{l=1}^p \frac{1}{l!} \Big(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} (u^{p-l}|_{\Gamma^-}) \Big) \right),$$

$$= \xi S_m \int_{-1}^0 \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}\eta + (-1)^p \sigma_e \partial_n u^p|_{\Gamma^+}.$$

This leads to the following Neumann transmission condition for u^p:

$$[\tilde{\sigma}\partial_n u^p]_\Gamma = \sigma_c \sum_{l=1}^{p} \frac{1}{l!}\Big(R_N^{l+1}(\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1}(u^{p-l}|_{\Gamma^-})\Big)$$
$$- (-1)^p \xi S_m \int_{-1}^{0} \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}\eta.$$

To obtain the Dirichlet transmission condition, we use the Taylor equality

$$\forall \eta \in (-1,0), \quad u_m^p(\cdot,\eta) = u_m^p|_{\eta=0} + \eta \partial_\eta u_m^p|_{\eta=0} + \int_0^\eta (\eta - s)\partial_\eta^2 u_m^p(\cdot,s)\mathrm{d}s.$$

According to (31)–(33)–(28)–(29) and applying the above equality with $\eta = -1$, we infer

$$\xi S_m\, [u^p]_\Gamma - \sigma_e \partial_n u^p|_{\Gamma^+} = \xi S_m \sum_{l=1}^{p} \frac{1}{l!} \left(R_N^l(\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l(u^{p-l}|_{\Gamma^-})\right)$$
$$+ (-1)^p \xi S_m \int_{-1}^{0} (1+\eta) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) \mathrm{d}\eta.$$

Therefore, we have formally obtain the asymptotic coefficients of the electric potential given by Theorem 1.3.

4 Preliminary estimates

Observe that the elementary problems of the expansion are non standard since they involve particular transmission conditions: according to equation (12c) the potential is not continuous but it is given by the flux itself and the parameter ξ. Since we aim at deriving uniform estimates with respect to $\xi \to 0$, in this section we present estimates for a generic problem similar to the elementary problems.

Let $s \geq 0$. Let g, g_N, and g_D belong respectively to $H^{1/2+s}(\partial\Omega)$, $H^{-1/2+s}(\Gamma)$, and to $H^{1/2+s}(\Gamma)$. Let v satisfy

$$\Delta v = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad v|_{\partial\Omega} = g, \tag{34a}$$
$$[\tilde{\sigma}\partial_n v]_\Gamma = -\xi g_N, \tag{34b}$$
$$\xi\, [v]_\Gamma - \frac{\sigma_e}{S_m}\partial_n v|_{\Gamma^+} = \xi g_D. \tag{34c}$$

4.1 Variational formulation

The variational formulation of Problem (34) is

$$\begin{cases} \text{Find } v \in g + PH_0^1(\Omega) \text{ such that for all } \phi \in PH_0^1(\Omega) \\ \displaystyle\int_\Omega \tilde{\sigma}\nabla v \nabla\overline{\phi}\mathrm{d}x + \xi \int_\Gamma S_m [v]_\Gamma [\overline{\phi}]_\Gamma \mathrm{d}s = \xi \int_\Gamma S_m g_D [\overline{\phi}]_\Gamma \mathrm{d}s + \xi \int_\Gamma g_N \overline{\phi}|_{\Gamma^-} \mathrm{d}s. \end{cases}$$

Observing that the bilinear form $a(\cdot,\cdot)$ defined by

$$\forall(\phi,\psi)\in\left[PH^1(\Omega)\right]^2,\quad a(\phi,\psi)=\int_\Omega \nabla\phi\nabla\overline{\psi}\mathrm{d}x+\int_\Gamma[\phi]_\Gamma[\overline{\psi}]_\Gamma\mathrm{d}s,$$

is an hermitian product on $PH^1(\Omega)$ we infer the following proposition.

Proposition 4.1. *Let $s\geq 0$. Let g, g_N, and g_D belong respectively to $H^{1/2+s}(\partial\Omega)$, $H^{-1/2+s}(\Gamma)$, and to $H^{1/2+s}(\Gamma)$. There exists a unique solution v to problem* (34)*, which belongs to $PH^1(\Omega)$. More precisely, using elliptic regularity*

$$v|_{\mathcal{O}_e}\in H^{1+s}(\mathcal{O}_e),\quad \textit{and}\quad v|_{\mathcal{O}_c}\in H^{2+s}(\mathcal{O}_c).$$

4.2 Uniform estimates in ξ for the generic problem

In this section, we aim at providing uniform estimates for ξ tending to zero, for the solution v to (34). Let $\mathcal{I}$ be defined by

$$\begin{cases}\Delta\mathcal{I}=0,\quad \text{in } \mathcal{O}_e,\\ \mathcal{I}|_{\partial\Omega}=0,\quad \mathcal{I}|_\Gamma=1.\end{cases}$$

Multiply (34) by $\mathcal{I}$ in $\mathcal{O}_e$ and by 1 in $\mathcal{O}_c$, and integrate by parts twice in $\mathcal{O}_e$, and once in $\mathcal{O}_c$. We obtain the following equality:

$$\int_\Gamma \sigma_e v|_{\Gamma^+}\partial_n\mathcal{I}|_{\Gamma^+}\mathrm{d}s=\int_{\partial\Omega}\sigma_e g\partial_n\mathcal{I}\mathrm{d}s+\xi\int_\Gamma g_N\mathrm{d}s.$$

Using the above equality, and multiplying (34) by $\mathcal{I}$ in $\mathcal{O}_e$ and by 0 in $\mathcal{O}_c$ integration by parts leads to

$$\int_\Gamma[v]_\Gamma\mathrm{d}s=\int_\Gamma\left(g_D+\frac{1}{S_m}g_N\right)\mathrm{d}s.$$

We therefore define the sequence $(v_e^l,v_c^l)_{l\geq 0}$ by

$$\begin{cases}\Delta v_e^0=0,\quad \text{in } \mathcal{O}_e,\\ v_e^0|_{\partial\Omega}=g,\quad \sigma_e\partial_n v_e^0|_\Gamma=0,\end{cases}$$

and

$$v_c^0=\frac{1}{\mathrm{mes}(\Gamma)}\left(\int_\Gamma v_e^0\mathrm{d}s-\int_\Gamma\left(g_D+\frac{1}{S_m}g_N\right)\mathrm{d}s\right),$$

and for $l\geq 1$

$$\begin{cases}\Delta v_e^l=0,\quad \text{in } \mathcal{O}_e,\\ v_e^l|_{\partial\Omega}=0,\quad \sigma_e\partial_n v_e^l|_\Gamma=S_m(v_e^{l-1}-v_c^{l-1})|_\Gamma-S_m\delta_1^l g_D,\end{cases}\tag{35a}$$

and

$$\begin{cases}\Delta v_c^l=0,\quad \text{in } \mathcal{O}_c,\\ \sigma_c\partial_n v_c^l|_\Gamma=S_m(v_e^{l-1}-v_c^{l-1})|_\Gamma-\delta_1^l\left(S_m g_D-g_N\right),\\ \int_\Gamma v_c^l\mathrm{d}s=\int_\Gamma v_e^l\mathrm{d}s.\end{cases}\tag{35b}$$

Proposition 4.2. *For all $k \geq 0$, $(v_e^k, v_c^k) \in H^{1+s+k}(\mathcal{O}_e) \times H^{2+s+k}(\mathcal{O}_c)$, and v_c^0 is constant. Moreover there exists a constant $C_{\mathcal{O}_e,\mathcal{O}_c}$, which only depends on the domain $\mathcal{O}_e$ and $\mathcal{O}_c$ such that for all $k \geq 0$*

$$\begin{aligned}\|v_e^k\|_{H^{1+s}(\mathcal{O}_e)} + \|v_c^k\|_{H^{2+s}(\mathcal{O}_c)} \leq C^k_{\mathcal{O}_e,\mathcal{O}_c}\Big(&|g|_{H^{1/2+s}(\partial\Omega)} \\ &+ |g_D|_{H^{1/2+s}(\Gamma)} + |g_N|_{H^{-1/2+s}(\Gamma)}\Big).\end{aligned} \tag{36}$$

Proof. According to (35) there exists a constant $C_{\mathcal{O}_e,\mathcal{O}_c}$ such that for all $k \geq 1$

$$\|v_e^{k+1}\|_{H^1(\mathcal{O}_e)} + \|v_c^{k+1}\|_{H^1(\mathcal{O}_c)} \leq C_{\mathcal{O}_e,\mathcal{O}_c}(\|v_e^k\|_{H^1(\mathcal{O}_e)} + \|v_c^k\|_{H^1(\mathcal{O}_c)}),$$

hence

$$\begin{aligned}\|v_e^{k+1}\|_{H^1(\mathcal{O}_e)} + \|v_c^{k+1}\|_{H^1(\mathcal{O}_c)} &\leq C^k_{\mathcal{O}_e,\mathcal{O}_c}(\|v_e^1\|_{H^1(\mathcal{O}_e)} + \|v_c^1\|_{H^1(\mathcal{O}_c)}), \\ &\leq C^{k+1}_{\mathcal{O}_e,\mathcal{O}_c}\Big(|g|_{H^{1/2}(\partial\Omega)} \\ &\quad + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)}\Big).\end{aligned}$$

Then we use the elliptic regularity satisfied by v_e^k and v_c^k to conclude. □

Proposition 4.3. *For ξ small enough, the series $\sum_{l\geq 0} \xi^l v_e^l$ and $\sum_{l\geq 0} \xi^l v_c^l$ normally converge in $H^{1+s}(\mathcal{O}_e)$ and in $H^{1+s}(\mathcal{O}_c)$ respectively. Moreover, the sums satisfy Problem (34), hence by uniqueness*

$$v|_{\mathcal{O}_e} = \sum_{l\geq 0} \xi^l v_e^l, \quad v|_{\mathcal{O}_c} = \sum_{l\geq 0} \xi^l v_c^l.$$

Particularly, since v_c^0 is constant, we infer that all the derivatives of $v|_{\mathcal{O}_c}$ are of order ξ.

Proof. Choose $\xi \in (0, 1/(2C_{\mathcal{O}_e,\mathcal{O}_c}))$ and Proposition 4.2 leads to Proposition 4.3. □

4.3 A priori estimates for the asymptotic coefficients

The following proposition is an application of both propositions 4.2–4.3. The proof performed using the induction process is left to the reader.

Proposition 4.4. *Let $n \in \mathbb{N}$ and $s \geq 0$ and suppose that $g \in H^{1/2+s+n}(\partial\Omega)$. The functions defined by (13)–(12) are uniquely determined and satisfy the following regularity for $0 \leq p \leq n$*

$$g_D^p \in H^{1/2+s+n-p}(\Gamma), \quad g_N^p \in H^{-1/2+s+n-p}(\Gamma), \tag{37}$$

$$u^p|_{\mathcal{O}_e} \in PH^{1+s+n-p}(\mathcal{O}_e), \quad u^p|_{\mathcal{O}_c} \in PH^{2+s+n-p}(\mathcal{O}_e), \tag{38}$$

$$u_m^p \in \mathscr{C}^\infty\Big((-1,0); H^{1/2+s+n-p}(\Gamma)\Big). \tag{39}$$

In addition there exists a constant $C_{\Omega,n}$ such that for all $0 \leq p \leq n$ the following estimates hold

$$\begin{aligned}\|u_e^p\|_{H^{1+s+n-p}(\mathcal{O}_e)} &\leq C_{\Omega,n}|g|_{H^{1/2+s+n}(\partial\Omega)}, \\ \|u_c^p\|_{H^{2+s+n-p}(\mathcal{O}_c)} &\leq C_{\Omega,n}|g|_{H^{1/2+s+n}(\partial\Omega)}, \\ \max_{\eta\in[-1,0]} |u_m^p(\cdot,\eta)|_{H^{1/2+s+n-p}(\Gamma)} &\leq C_{\Omega,n}|g|_{H^{1/2+s+n}(\partial\Omega)}.\end{aligned}$$

Moreover, we have the following estimate uniformly in ξ:

$$\|\nabla u_c^l\|_{H^{1+s+n-p}(\mathcal{O}_c)} \leq C_{\Omega,n}\xi |g|_{H^{1/2+s+n}(\partial\Omega)}.$$

Remark 4.5. *More precisely, according to (12) for all $p \geq 0$ there exists a sequence of functions $(f^q)_{0\leq q\leq p}$ defined on Γ such that in the cylinder $\Gamma \times (-1,0)$, the coefficients u_m^p writes*

$$\forall(\mathbf{x_T},\eta) \in \Gamma \times (-1,0), \quad u_m^p(\mathbf{x_T},\eta) = \sum_{q=0}^{p} \eta^q f^q(\mathbf{x_T}).$$

The functions $(f^q)_{0\leq q\leq p}$ are defined by induction using (12) and they satisfy the following regularity:

$$f^q \in H^{1/2+s+p-q}(\Gamma), \quad \textit{for } q \in 0,\cdots,p.$$

5 Error estimates

For $p \geq 1$, we denote by $\mathcal{L}_p$ the operator on $\Gamma \times (-1,0)$ of order 1 in the η-variable and 2 in the $\mathbf{x_T}$-variables defined as

$$\mathcal{L}_p = \Delta_g - \frac{1}{\delta^2}\left(\partial_\eta^2 + a_{33}^0\delta\partial_\eta + \sum_{l=2}^{p}\delta^l \frac{\eta^{l-2}}{(l-2)!}\left(\frac{\eta}{l-1}a_{33}^{l-1}\partial_\eta + \mathcal{S}_\Gamma^{l-2}\right)\right).$$

For any function $u \in C^\infty([-1,0]; H^{s+3/2}(\Gamma))$

$$\max_{\eta\in(-1,0)} |\mathcal{L}_p(u)(\cdot,\eta)|_{H^{-1/2+s}\left(\Gamma\right)} \leq C_p\delta^{p-1}\max_{\eta\in[0,1]}\left(|u(\cdot,\eta)|_{H^{s+3/2}(\Gamma)} + |\partial_\eta u(\cdot,\eta)|_{H^{s+3/2}(\Gamma)}\right). \tag{40}$$

According to Proposition 4.3, for all $k \geq 0$, the derivatives of $u^k|_{\mathcal{O}_c}$ are of order ξ. Define U^k as follows:

$$U^k = \begin{cases} u^0 - \delta u^1 + \cdots + (-\delta)^k u^k, \text{ in } \mathcal{O}_e \cup \mathcal{O}_c^\delta, \\ u_0^m + \delta u_1^m + \cdots + \delta^k u_k^m, \text{ in } \Gamma \times (-1,0). \end{cases}$$

Our main result provides uniform estimates of the error performed by the approximate expansion at any order, for $|\xi|$ tending to zero. The following theorem straightforwardly leads to Theorem 1.3.

Theorem 5.1. *Let $k \geq 0$. Suppose that g belongs to $H^{5/2+k}(\Gamma)$. Let $w^k = u - U^k$.*

There exists a constant C_k such that

$$\|w^k\|_{H^1(\mathcal{O}_e)} \leq C_k\xi\delta^{k+1}|g|_{H^{5/2+k}(\partial\Omega)}, \tag{41}$$

and for all ω compactly embedded in $\mathcal{O}_c$

$$\|w^k\|_{H^1(\omega)} \leq C_k\delta^{k+1}|g|_{H^{5/2+k}(\partial\Omega)}, \tag{42}$$

$$\|\nabla w^k\|_{L^2(\omega)} \leq C_k\xi\delta^{k+1}|g|_{H^{5/2+k}(\partial\Omega)}. \tag{43}$$

Proof. By definition of the functions $(u^l)_{l\geq 0}$ and $(u^m_l)_{l\geq 0}$, we infer

$$\sigma\Delta w^k = 0, \text{ in } \mathcal{O}_e \cup \mathcal{O}_c^\delta,$$
$$\sigma_m\Delta_g w^k = -\sigma_m \mathcal{L}_k(U^k), \text{ in } \Gamma\times(-1,0),$$
$$w^k|_{\partial\Omega} = 0,$$

with the following transmission conditions

$$\sigma_c\partial_n w^k|_{\Gamma^-_{-\delta}} - \sigma_m\partial_n w^k|_{\Gamma^+_{-\delta}} = \sigma_m\partial_n U^k|_{\Gamma^+_{-\delta}} - \sigma_c\partial_n U^k|_{\Gamma^-_{-\delta}},$$
$$\sigma_m\partial_n w^k|_{\Gamma^-} - \sigma_e\partial_n w^k|_{\Gamma^+} = 0,$$
$$w^k|_{\Gamma^-_{-\delta}} - w^k|_{\Gamma^+_{-\delta}} = U^k|_{\Gamma^+_{-\delta}} - U^k_{\Gamma^-_{-\delta}},$$
$$w^k|_{\Gamma^+} - w^k|_{\Gamma^-} = 0.$$

Moreover, by definition we have

$$\sigma_m\partial_n U^k|_{\Gamma^+_{-\delta}} - \sigma_c\partial_n U^k|_{\Gamma^-_{-\delta}} = -\sigma_c\delta^{k+1}T^{k+2}(U^k),$$
$$U^k|_{\Gamma^+_{-\delta}} - U^k|_{\Gamma^-_{-\delta}} = -\delta^{k+1}T^{k+1}(U^k).$$

According to Proposition 4.4 and to Corollary 2.2 we infer

$$\left|T^{k+1+i}(U^k)\right|_{H^{1/2-i}(\Gamma)} \leq C\xi, \quad i = 0,1.$$

Multiply by $\overline{\phi}$ in $H^1(\Omega)$ and integrate by parts with the help of Green's formula. Using equality (40) and the metric (g_{ij}) in local coordinates in the layer, by taking $\phi = w^k$ and using the trace theorem on Γ we infer that

$$\|\sigma\nabla w^k\|_{L^2(\Omega)} = O(\xi\delta^{k+1/2}).$$

Then defining $\tilde{w}^k = w^k - \delta^{k+1}u^m_{k+1}$ in $\Gamma\times(-1,0)$ and $\tilde{w}^k = w^k$ in $\mathcal{O}_e^\delta\cup\mathcal{O}_c$ we infer

$$\|\sigma\nabla\tilde{w}^k\|_{L^2(\Omega)} = O(\xi\delta^{k+1}),$$

hence the theorem.

□

6 Numerical simulations for a 3D–resistive thin layer

6.1 2D simulations by Fourier expansion

We first start with bidimensional simulation. Domains Ω and $\mathcal{O}_i$ are two disks centered in 0 of respective radius equal to 2 and 1. Parameters σ_c and S_m are taken equal to 1 and σ_e to 2. Function g is $g(\theta) = \cos(3\theta)$. Observe that in such a case, the mean curvature of the infinite cylinder equals 1/2, hence $\mathscr{H}$ of equation (21) simply equals 1/2.

From the analytic solutions the expected rates of convergence with δ are verified since the numerical estimated exponents are respectively 1 for the order

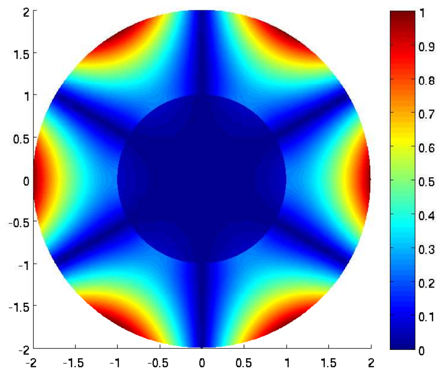


Figure 2: Real part of $u^0 - \delta u^1$ computed by the finite element method with $\delta = 0.05$.

0 and 2 for the order 1; see Fig. 3. The green line corresponds to the zeroth order for the soft contrast media. In the soft contrast case, the effect of the layer appears only at the first order term, meaning that the influence of the layer is of order δ. We observe that this potential does not provide an accurate approximation of the potential, since the error estimate does not converge, when δ tends to zero.

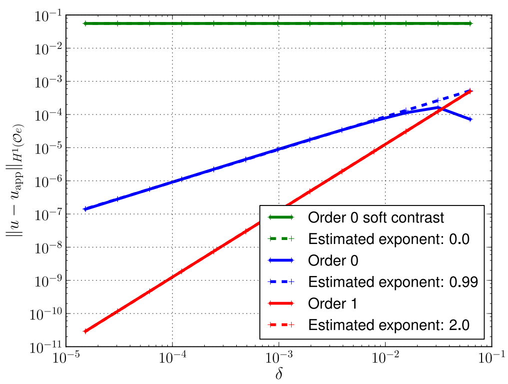


Figure 3: Log-log diagram of the error versus the membrane thickness.

As predicted by Theorem 5.1 and Theorem 1.3, the rate of convergence of the zeroth order term u^0 defined by (20) increases when the membrane conductivity decreases, justifying the use of such approximate conditions whatever small the membrane conductivity is; see Fig. 4. This behavior is also recovered when using the finite element method based on the variational formulation (35) for solving our model problem; see Fig. 5.

We conclude by observing that the convergence rates are asymptotic meaning that if the membrane is not thin enough, it is not sure that our approximate transmission conditions are relevant. For instance, in the configurations we chose, the accuracy of the transmission conditions seems to increase for a thickness around 10^{-1} but if we were choosing other configurations, the accuracy could have decreased as well. Therefore, the shape of the error curves are less significant for a thickness larger than 10^{-2}.

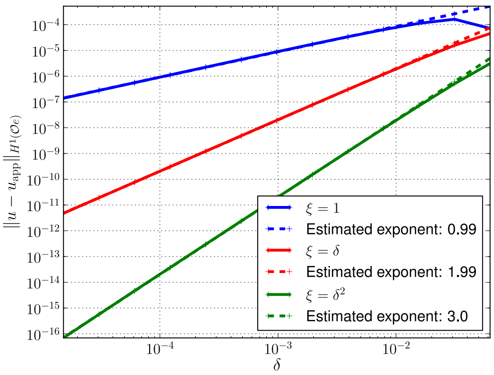


Figure 4: Log-log diagram of the error in $\mathcal{O}_e$ of the zeroth order for different small conductivities.

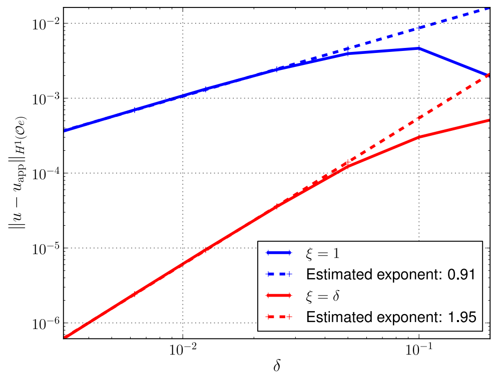


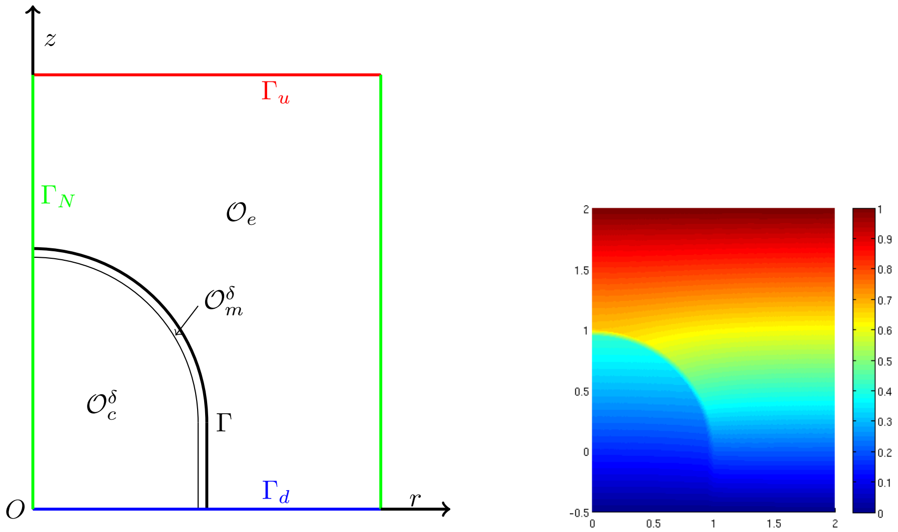
Figure 5: Log-log diagram of the error in $\mathcal{O}_e$ of the zeroth order for different small conductivities. Computations are performed with the finite element method.

6.2 3D simulations by the finite element method

For this simple 3D example, the geometry and the source term are supposed to be axisymmetric (independent of the angular variables in a cylindrical coordinate system). It enables to perform the computations on an equivalent 2D problem. The reduced 2D computational domain is given on Fig. 6(a) and the solution u^δ satisfies the following reduced problem:

$$\begin{cases} \dfrac{\partial}{\partial r}\left(\sigma r \dfrac{\partial u^\delta}{\partial r}\right) + \dfrac{\partial}{\partial z}\left(\sigma r \dfrac{\partial u^\delta}{\partial z}\right) = 0, \\ u^\delta = 1, \text{ on } \Gamma_u \text{ and } u^\delta = 0, \text{ on } \Gamma_d, \\ \dfrac{\partial u^\delta}{\partial n} = 0, \text{ on } \Gamma_N. \end{cases} \tag{44}$$

Moreover, for sake of simplicity, ξ equals to h and σ_c, σ_e and S_m are equal to 1.



(a) The computational domain. (b) Solution u^δ for $\delta = 0.05$.

Figure 6: Computational domain and the solution for one particular δ.

As no analytic solution is known for this example, the reference solution is computed by the finite element method. On Fig. 6(b), this solution is given for $\delta = 0.05$. The asymptotic convergence in δ is obtained for the order 0 and in δ^2 for the order 1 as shown on Fig. 7. In this particular situation, the pre-asymptotic regime is shorter than in the 2D example.

7 Concluding remarks

In this paper, we have provided asymptotic expansions of the electric potential for a resistive thin layer, whatever small the membrane conductivity is. The expansion leads to particular elementary problems in the sense that the jump of the potential across the membrane is given by the flux. We have provided a variational formulation that allows to compute the potential without any lifting of the flux, allowing to compute the approximate potential in only one computation whatever small the membrane conductivity.

Such results are “important” when dealing with time dependent problem. More precisely, the time dependent electric potential in a biological cell satisfies

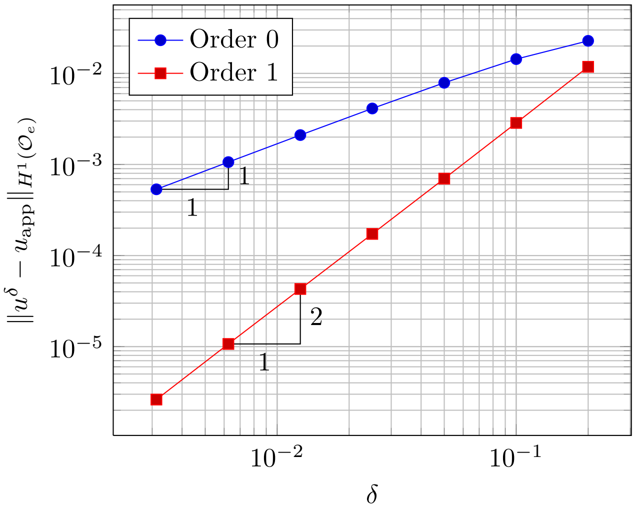


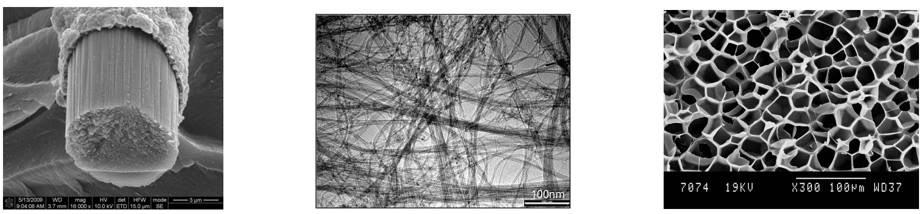
Figure 7: Log-log diagram of the error versus the membrane thickness.

(see part 5 of [15])

$$\begin{cases} \Delta V = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad V(t,x)|_{x\in\partial\Omega} = g, \quad V(0,\cdot) = 0 \\ C_m \partial_t \left[V\right]_\Gamma + S_m \left[V\right]_\Gamma - \sigma_e \partial_n u^{approx}|_{\Gamma^+} = 0 \\ \left[\tilde{\sigma}\partial_n V\right]_\Gamma = 0, \end{cases}$$

Guyomarc'h and Lee propose a discontinuous Galerkin method to compute V by making explicit the flux. The main drawback of their time explicit scheme is that a Courant-Friedrich-Lewy type condition, that links the time-step and the mesh, has to be satisfied leading to time– and resource– consuming computations. Our variational formulation that can be easily computed by discontinuous Galerkin method should lead to unconditionally stable scheme since the flux is implicited.

Another important application of our present paper holds for magnetic properties of composite materials composed of foams with carbon nanotubes (CNTs) trapped into the wall of bubbles (see Fig.8). The magnetic permeability of the



(a) Carbon Microfiber (b) Carbone Nanotubes (CNT) (c) Polymer foams with CNTs

Figure 8: Description of polymer foams with CNT at different scales

CNTs is very high (depending on the concentration of CNT) compared with the air permeability μ_0, but the thickness of the bubble wall is thin compared with the bubble diameter. The domain Ω is composed of a domain $\mathcal{O}_0$ with a network $\mathcal{O}_\delta$ containing the CNTs. The magnetostatic equation writes:

$$-\nabla.\left(\frac{1}{\mu_\delta}\nabla H_\delta\right) = H_{source}, \quad \text{in } \Omega$$
$$H_{\delta|\partial\Omega} = 0.$$

Due to the high contrast in the permeabilities and since the thickness of the CNT network is small, the computation of the magnetostatic field is time- and resource- consuming, especially in three-dimensions.

Our result shows that, denoting the CNT permeability by

$$\mu_\delta|_{\Omega_\delta} = \delta^{-1}\mu_m,$$

the magnetostatic field H_δ can be approached at the order δ by H_0, the variational solution in $PH_0^1(\Omega)$ to:

$$\frac{1}{\mu_0}\int_\Omega \nabla H_0 \nabla\phi \mathrm{d}x + \frac{\mu_0}{\mu_m}\int_\Gamma [H_0]\,[\phi]\,\mathrm{d}s = \int_\Omega H_{source}\phi \mathrm{d}x, \quad \forall\phi \in PH_0^1(\Omega),$$

where Γ is the bidimensional surface corresponding to the network of CNT.

Acknowledgments

The authors thank warmly P. Dular and R. Sabariego from the Electrical Engineering Department of the University of Liège, and L. Krähenbühl from the Laboratoire Ampère of Lyon for helpful discussions for the motivations of the work presented here. The authors are also very grateful to K.Schmidt from DFG Research Center for well-considered advices and suggestions. C.P. is granted by the ANR INTCELL.

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ISSN 0249-6399